

# Quadratic Deformations of Lie-Poisson Structures \*

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## Abstract

In this letter, first we give a decomposition for any Lie-Poisson structure  $\pi_g$  associated to the modular vector. In particular,  $\pi_g$  splits into two compatible Lie-Poisson structures if  $\dim g \leq 3$ . As an application, we classified quadratic deformations of Lie-Poisson structures on  $\mathbb{R}^3$  up to linear diffeomorphisms.

## 1 Introduction

It is known that linear and quadratic Poisson structures are two most basic and important Poisson structures both for their rich algebraic and geometric properties and various applications in physics and other fields of mathematics. The linear Poisson structures are in one-to-one correspondence with Lie algebra structures and usually called Lie-Poisson structures. The idea of using linear Poisson brackets to understand the structure of Lie algebras can be traced back to the work of Lie. In this spirit there have been some suggestions of pursuing this geometric approach for Lie algebra structures (e.g., see [2], [8] and [13]). Besides the linear Poisson structures, quadratic Poisson structures are also generally studied from various aspects (e.g., [5], [6], [12] and see [15] for more comments).

The purpose of this letter is to study quadratic deformations of Lie-Poisson structures and their classification on  $\mathbb{R}^3$ . Main motivation for us comes from the work in [1] and [3], where some special quadratic deformation of a Lie-Poisson structure appears when the authors study some geometric objects such as holonomy and symplectic connection. In [4] a simple example is also shown as an example of Poisson-Dirac submanifolds. On the other hand, in [6] and [11], the quadratic Poisson structures on  $\mathbb{R}^3$ , which can be considered as quadratic deformations of the abelian Lie-Poisson

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structure, are totally classified. Thus, it is natural to consider all possible quadratic deformations for any Lie-Poisson structure. To classify quadratic deformations for a fixed Lie-Poisson structure it is enough to use linear transformations, which keep the degree of a homogenous tensor field. The classification under general diffeomorphism is more complicated. Please see [7] for the classification under local diffeomorphisms of every 3-dim Poisson structure vanishing at a point with a non-zero linear part.

To save space we just write out details of quadratic deformations for some spacial cases. The others can be done by same way without any difficulty except for some tedious computations.

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## 2 The Classification of Lie-Poisson structures on $\mathbb{R}^3$

Let  $\Omega = dx_1 \wedge dx_2 \cdots \wedge dx_n$  be the canonical volume form on  $\mathbb{R}^n$ . Then  $\Omega$  induces an isomorphism  $\Phi$  from the space of all  $i$ -multiple vector fields to the space of all  $(n-i)$ -forms. Let  $d$  denote the usual exterior differential on forms and

$$D = (-1)^{k+1} \Phi^{-1} \circ d \circ \Phi: \mathcal{X}^k(\mathbb{R}^n) \rightarrow \mathcal{X}^{k-1}(\mathbb{R}^n),$$

its pull back under the isomorphism  $\Phi$ . The Schouten bracket can be written in terms of this operator as follows[9]:

$$[U, V] = D(U \wedge V) - D(U) \wedge V - (-1)^i U \wedge D(V), \quad (1)$$

for all  $U \in \mathcal{X}^i(\mathbb{R}^n)$  and  $V \in \mathcal{X}^j(\mathbb{R}^n)$ . It is obvious that there is a one-to-one correspondence between matrices in  $\mathfrak{gl}(n)$  and linear vector fields on  $\mathbb{R}^n$ , i.e.,

$$A = (a_{ij}) \longleftrightarrow \hat{A} = \sum_{ij} a_{ij} x_j \frac{\partial}{\partial x_i}, \quad \text{div}_\Omega \hat{A} = D(\hat{A}) = \text{tr} A. \quad (2)$$

A vector  $k \in \mathbb{R}^n$  corresponds a constant vector field  $\hat{k}$  by translation and satisfies

$$\text{div}_\Omega \hat{k} = D(\hat{k}) = 0, \quad [\hat{A}, \hat{k}] = -\hat{A}k, \quad \forall A \in \mathfrak{gl}(n). \quad (3)$$

For a given Poisson tensor  $\pi$ , let  $D(\pi)$  be its modular vector field (see [14], which is also called the curl vector field in [6]). Such a vector field is always compatible with  $\pi$ , i.e.,  $[D(\pi), \pi] = 0$ . A Poisson structure is called unimodular if  $D(\pi) = 0$ . By (3), it is easy to see that, for any  $k \in \mathbb{R}^n$ , The bi-vector field  $\hat{I} \wedge \hat{k}$  is a linear

Poisson structure with the modular vector  $(n-1)k$ , where  $I$  is the identity matrix. The corresponding Lie algebra is called *book algebra* when  $n = 3$  and  $k = (0, 0, 1)$ . Next we give a similar decomposition for Lie-Poisson structures as doing in [11] for quadratic Poisson structures.

**Theorem 2.1.** *Any linear Poisson structure  $\pi$  on  $\mathbb{R}^n$  has a unique decomposition:*

$$\pi = \frac{1}{n-1} \hat{I} \wedge \hat{k} + \Lambda, \quad (4)$$

where  $k \in \mathbb{R}^n$  is the modular vector of  $\pi$  and  $\Lambda$  is a linear bi-vector field such that  $D(\Lambda) = 0$ .  $\Lambda$  is a Poisson structure compatible with  $\pi$  if and only if  $D(\Lambda \wedge \Lambda) = 0$ . In particular,  $\Lambda$  is always a unimodular Lie-Poisson structure on  $\mathbb{R}^3$ .

**Proof.** Let  $\hat{k} = D(\pi)$  and define a bi-vector field  $\Lambda =: \pi - \frac{1}{n-1} \hat{I} \wedge \hat{k}$ . Then  $D(\Lambda) = 0$  and  $[\Lambda, \Lambda] = D(\Lambda \wedge \Lambda)$  by formula (1). The fact that  $[\hat{k}, \Lambda] = 0$  is because  $\hat{k}$  is the modular vector field for both Poisson structures  $\pi$  and  $\hat{I} \wedge \hat{k}$ . Finally, by (1) and the fact that  $[\hat{I}, \pi] = -\pi$ , one can see that  $\Lambda$  is compatible with  $\pi$  if  $\Lambda$  is a Poisson structure. ■

From now on, we focus on the case that  $n = 3$ . In this case there exists a unique homogeneous quadratic 3-vector field  $L$  such that  $\Lambda = D(L)$  since  $D(\Lambda) = 0$  and the cohomology groups here are trivial. The quadratic 3-vector fields are in one-to-one correspondence with quadratic functions  $f$  via the volume form  $\Omega$ . We use  $\pi_f$  to denote this unimodular linear Poisson structure which is given by

$$\pi_f = \Phi^{-1}(df) = \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}. \quad (5)$$

The compatibility condition with the modular vector field  $\hat{k}$  is given by

$$[\hat{k}, \pi_f] = \Phi^{-1} L_{\hat{k}}(df) = \Phi^{-1} d(\hat{k}f) = 0 \iff \hat{k}f = 0. \quad (6)$$

Therefore, the linear Poisson structures on  $\mathbb{R}^3$  are in one-to-one correspondence with pairs  $(k, f)$ , where  $k$  is a vector and  $f$  is a quadratic function, such that  $\hat{k}f = 0$ . As doing in [11] for the quadratic Poisson structures, here we also call such a pair  $(k, f)$  as a **compatible pair** of the corresponding linear Poisson structure  $\pi$  and usually denote  $\pi = \pi_{k,f}$ . The next result characterizes the isomorphisms of linear Poisson structures by means of their compatible pairs.

**Theorem 2.2.** *Let  $\pi_1$  and  $\pi_2$  be two linear Poisson structures on  $\mathbb{R}^3$  determined by compatible pairs  $(k_1, f_1)$  and  $(k_2, f_2)$  respectively. Then  $\pi_1$  is isomorphic to  $\pi_2$  if and only if there is a  $T \in GL(3)$  such that*

$$k_2 = Tk_1, \quad f_2 = \det(T) f_1 \circ T^{-1}.$$

Particularly, the automorphism group of  $\pi_{k,f}$ , denoted by  $Aut(\pi_{k,f})$ , is

$$Aut(\pi_{k,f}) = \{T \mid T \in GL(3), Tk = k, f \circ T = \det(T)f\}. \quad (7)$$

Consequently, for the corresponding Lie algebra  $\mathfrak{g}$  of  $\pi_{k,f}$ , one has

$$Der(\mathfrak{g}) \cong \{D \mid D \in \mathfrak{gl}(3), Dk = 0, \hat{D}f = (\text{tr}D)f\}. \quad (8)$$

**Proof.**  $T$  is an isomorphism means that  $T_*\pi_1 = \pi_2$ , by Theorem 2.1 and properties of the modular vector, which is equivalent to that  $Tk_1 = k_2$  and  $T_*\pi_{f_1} = \pi_{f_2}$ , which is equivalent to  $|T|f_1 = f_2 \circ T$ . The other conclusions are easy to be checked. ■

Moreover, it will be seen that the traceless derivations,

$$Der_0(\mathfrak{g}) \cong \{D \mid D \in \mathfrak{sl}(3), Dk = 0, \hat{D}f = 0\}, \quad (9)$$

play an important role for the quadratic deformation. As an application of Theorems 2.1 and 2.2, we show a simple way to classify linear Poisson structures on  $\mathbb{R}^3$ .

**Theorem 2.3.** Any linear Poisson structure  $\pi_{k,f}$  on  $\mathbb{R}^3$  is isomorphic to one of the following standard forms:

<p>(A). <math>k = 0</math> (unimodular )</p> <p>(1). <math>f = 0</math></p> <p>(2). <math>f = x^2 + y^2 + z^2</math></p> <p>(3). <math>f = x^2 + y^2 - z^2</math></p> <p>(4). <math>f = x^2 + y^2</math></p> <p>(5). <math>f = x^2 - y^2</math></p> <p>(6). <math>f = x^2</math></p>	<p>(B). <math>k = (0, 0, 1)^T, \hat{k} = \frac{\partial}{\partial z}</math>,</p> <p>(7). <math>f = 0</math></p> <p>(8). <math>f = a(x^2 + y^2), a &gt; 0</math></p> <p>(9). <math>f = a(x^2 - y^2), a &gt; 0</math></p> <p>(10). <math>f = x^2</math>.</p>
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**Proof.** For unimodular cases,  $\pi_{f_1}$  is isomorphic to  $\pi_{f_2}$  if and only if there is some  $T \in GL(3)$  such that  $f_2 \circ T = |T| f_1$  by Theorem 2.2. First we take a  $T' \in GL(3)$  such that the quadratic function  $f' = f \circ T'$  is one of the standard forms listed from (1) to (6). Assume  $T = \det(T')^{-1}T'$ , then we have  $f \circ T = |T| f'$ .

If  $k \neq 0$ , we can take  $k = (0, 0, 1)^T$  by a coordinate transformation. Then  $\exists A \in \text{symm}(2)$  such that  $f = (x, y)A(x, y)^T$  because  $\hat{k}f = \frac{\partial f}{\partial z} = 0$  by (6). Moreover,  $T$  must have the form:  $\begin{pmatrix} S & 0 \\ \alpha & 1 \end{pmatrix}$ , where  $S \in GL(2)$ ,  $\alpha \in \mathbb{R}^2$ , since  $Tk = k$  by Theorem 2.2. Thus it is seen that, for any  $T$  having the form above and  $f \neq 0$ , the induced new quadratic function from  $f$  must be determined by the matrix  $|S|^{-1}S^TAS$  so that its standard form can be fixed up to a constant. Furthermore it is easy to check that this

constant can be adjust by a sign and have the form that listed in the theorem. As in case (10), we can adjust  $S$  such that  $\det(S) = 1$ . ■

Denote  $\mathfrak{g}_i (i = 1, \dots, 10)$  the corresponding Lie algebra of linear Poisson structures  $\pi_i$ . Comparing with the classification described in [10] via the dimension of the derived algebra  $[\mathfrak{g}, \mathfrak{g}]$ , Case (1) is abelian. Cases (2) and (3) are simple Lie algebras. In cases (4), (5), (7)-(10) the derived algebra has dimension 2 except for  $a = 1/4$  in case (9). In case (6) and case (9) with  $a = 1/4$ , the derived algebra has dimension 1. The Lie algebras (8)-(10) are three twisted Lie algebras of  $\mathfrak{g}_7$  corresponding three 2-cocycles respectively.

By means of Theorem 2.2, we see that the automorphism groups are independent on the constant in Cases (8)-(9). Combining with Theorem 2.3, we can get the automorphism groups of 3-dim. linear Poisson structures.

**Theorem 2.4.** *Let  $G_i$ ,  $i = 1, \dots, 10$  denote the automorphism group of the linear Poisson structure which corresponds to Case (i) in Theorem 2.3. Then we have*

$$\begin{aligned} G_1 &= GL(3), G_2 = SO(3), G_3 = SO(2, 1), G_4 = \left\{ \begin{pmatrix} \lambda T & 0 \\ \xi & |T| \end{pmatrix} \mid T \in O(2) \right\}, \\ G_5 &= \left\{ \begin{pmatrix} \lambda \alpha & \beta & 0 \\ \lambda \beta & \alpha & 0 \\ \gamma & \delta & \lambda \end{pmatrix} \mid \lambda = \pm 1, \alpha^2 \neq \beta^2 \right\}, G_6 = \left\{ \begin{pmatrix} a & 0 \\ \xi & A \end{pmatrix} \mid |A| = a \right\}, \\ G_7 &= \left\{ \begin{pmatrix} A & 0 \\ \xi & 1 \end{pmatrix} \mid A \in GL(2) \right\}, G_8 = \left\{ \begin{pmatrix} \lambda T & 0 \\ \xi & 1 \end{pmatrix} \mid T \in SO(2), \lambda \neq 0 \right\}, \\ G_9 &= \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \alpha & 0 \\ \gamma & \delta & 1 \end{pmatrix} \mid \alpha^2 \neq \beta^2 \right\}, G_{10} = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & \delta & 1 \end{pmatrix} \mid \alpha \neq 0 \right\}. \end{aligned}$$

Note that  $G_8$ - $G_{10}$  are subgroups of  $G_4$ - $G_6$  respectively with  $Tk = k$  and subgroups of  $G_7$ . This fact will be used in the last section to classify quadratic deformations of Lie-Poisson structures of Cases (8)-(10).

### 3 Quadratic Deformations

In [11], any quadratic Poisson structure on  $\mathbb{R}^3$  is characterized by its compatible pair  $(K, F)$ , where  $K \in \mathfrak{sl}(3)$  and  $F$  is a homogeneous cubic polynomial such that  $\hat{K}F = 0$  and  $\pi_{K, F} = \pi_F + \frac{1}{3}\hat{F} \wedge \hat{K}$ . Such a Poisson structure can be considered as a quadratic deformation of the abelian Lie-Poisson structure.

For any Lie-Poisson structure  $\pi_{k,f}$ , we shall study its quadratic deformations  $\pi_{k,f} + \pi_{K,F}$ , evidently, which is still a Poisson structure if and only if  $[\pi_{k,f}, \pi_{K,F}] = 0$ . For convenience, denote by  $\mathcal{C}$  the compatible pairs of the quadratic Poisson structures and  $\mathcal{C}_{k,f}$  the compatible pairs that can make quadratic deformations of the linear Poisson structure  $\pi_{k,f}$ , i.e.,

$$\mathcal{C} = \{(K, F); K \in \mathfrak{sl}(3), F \text{ is a 3- polynomial s.t. } \hat{K}F = 0\}. \quad (10)$$

$$\mathcal{C}_{k,f} = \{(K, F) \in \mathcal{C}, [\pi_{k,f}, \pi_{K,F}] = 0\}. \quad (11)$$

Denote  $X = (x, y, z) \in \mathbb{R}^3$  and for any quadratic function  $f$ , write  $f = (AX, X)$ , where  $A$  is a symmetric matrix. For any  $k \in \mathfrak{g}^*$ , denote  $\tilde{k} : \mathfrak{g}^* \rightarrow \mathfrak{g}$  the skew-symmetric matrix corresponding to  $\Phi^{-1}(k) \in \mathfrak{g} \wedge \mathfrak{g}$ . Then we have

**Theorem 3.1.** *Let  $(k, f)$  and  $(K, F)$  be two compatible pairs of a Lie-Poisson structure and a quadratic Poisson structure on  $\mathbb{R}^3$  respectively. Then we have*

$$(K, F) \in \mathcal{C}_{k,f} \iff \hat{k}F = -\frac{1}{6}X(12A + \tilde{k})KX^T. \quad (12)$$

*Especially, in unimodular cases that  $k = 0$ , we have*

$$\mathcal{C}_{0,f} = \{(K, F) \in \mathcal{C}, \hat{K}f = 0\}, \text{ i.e., } K \in \text{Der}_0(\mathfrak{g}). \quad (13)$$

**Proof.** By means of Equality (1),  $\pi_{K,F}$  makes quadratic deformation of Lie-Poisson structure  $\pi_{k,f}$  if and only if

$$-[\pi_{k,f}, \pi_{K,F}] = \hat{k} \wedge \left(\frac{1}{3}\hat{I} \wedge \hat{K} + \pi_f\right) + \hat{K} \wedge \left(\frac{1}{2}\hat{I} \wedge \hat{k} + \pi_f\right) = 0.$$

This is just  $6\pi_{\hat{k}F + \hat{K}f} = D(\hat{K} \wedge \hat{k} \wedge \hat{I})$  and is equivalent to

$$6(\hat{k}F + \hat{K}f) = -(x, y, z)(\tilde{k}K)(x, y, z)^T.$$

Note that  $\hat{K}f = (x, y, z)(AK + (AK)^T)(x, y, z)^T$ , so a compatible pair  $(K, F) \in \mathcal{C}_{k,f}$  if and only if it satisfies Equality (12). ■

Next we consider the problem to classify the quadratic deformations of a Lie-Poisson structure on  $\mathbb{R}^3$ . We restrict us to the case that the linear parts of two isomorphic Poisson structures are also isomorphic. That is, we consider the classification of quadratic deformations for a fixed Lie-Poisson structure by using linear transformations, which keep the degree of a homogenous tensor field. The following theorem is straightforward.

**Theorem 3.2.** *Let  $(K_i, F_i)_{i=1,2} \in \mathcal{C}_{k,f}$ , where  $(k, f)$  is the compatible pair of Lie-Poisson structure  $\pi_{k,f}$ . Then  $\pi_{k,f} + \pi_{K_i, F_i}$  are isomorphic if and only if there is a  $T \in \text{Aut}(\pi_{k,f})$  such that*

$$K_2 = T K_1 T^{-1} \text{ and } F_2 = \det(T) F_1 \circ T^{-1}. \quad (14)$$

*Consequently, the classification of quadratic deformations of a Lie-Poisson structure  $\pi_{k,f}$  on  $\mathbb{R}^3$  is parameterized by the orbit space  $\text{Aut}(\pi_{k,f}) \backslash \mathcal{C}_{k,f}$ .*

It is known that  $GL(3)$  acts on the space of compatible pairs  $\mathcal{C}$  given in (10) as

$$T : \mathcal{C} \rightarrow \mathcal{C}, \quad T(K, F) =: (TKT^{-1}, \det(T)F \circ T^{-1}), \quad \forall T \in GL(3), \quad (15)$$

so that, as did in [11], the quadratic Poisson structures on  $\mathbb{R}^3$  are classified by the Adjoint orbits of  $GL(3)$  and one can take the Jordan forms as the standard forms. The above theorem shows that one should do more things on the orbits of the Jordan forms to classify the quadratic deformations of a Lie-Poisson structure.

Notice that  $\mathcal{C}_{k,f}$  is relatively easy to be determined in the unimodular cases but it is difficult in Cases (7) – (10) because the unknown data  $K$  and  $F$  are involved together in Equation (12). In these cases we should fix  $K$  firstly with some standard form and then to find all possible compatible 3-polynomials  $F$  satisfying Equation (12). For a fixed Jordan form  $K \in \mathfrak{sl}(3)$ , denote by  $G_K \subset GL(3)$  as its isotropy subgroup for the adjoint action and  $\mathcal{J}_K = GL(3)/G_K$  the adjoint orbit through  $K$ . Obviously,  $\mathcal{J}_K$  is invariant under the adjoint action of  $\text{Aut}(\pi_{k,f})$  and the orbit space is a double quotient space:

$$\text{Aut}(\pi_{k,f}) \backslash \mathcal{J}_K \cong \text{Aut}(\pi_{k,f}) \backslash (GL(3)/G_K) \quad (16)$$

Now we give a scheme to classify quadratic deformations of Lie-Poisson structures of Cases (7)-(10) as follows.

- (1) Take a standard form of Lie-Poisson structure  $\pi_{k,f}$  from the list in Theorem 2.3(B) and take a Jordan standard form  $K$ , for which the compatible 3-polynomial  $F$  was fixed in [11].
- (2) Choose a representative element  $\tilde{K}$  in each  $\text{Aut}(\pi_{k,f})$ -orbit in  $\mathcal{J}_K$ . This means that there exists a  $T \in GL(3)$  such that  $\tilde{K} = TKT^{-1}$ . Then we get a compatible pair  $(\tilde{K}, \tilde{F}) \in \mathcal{C}$  by Formula (14) from the known compatible pair  $(K, F)$ .
- (3) Check the compatible pair  $(\tilde{K}, \tilde{F})$  given above if it satisfies Equation (12).
- (4) In case that there are more than one cubic polynomials satisfying Equation (12), just take one representative element.

By Theorem 3.2, the Poisson structures  $\pi_{k,f} + \pi_{\tilde{K},\tilde{F}}$  classify quadratic deformations of the Lie-Poisson structure  $\pi_{k,f}$  after checking for all Jordan standard forms. In following sections we classify quadratic deformations of Lie-Poisson structures. Emphasizing again, to save space we just write out some cases with details. The others can be done by same way without any difficulty except for some tedious computations.

## 4 The unimodular cases

For Lie-Poisson structures (1)-(6) listed in Theorem 2.3, where their modular characters vanish,  $(K, F) \in \mathcal{C}_{0,f}$  if and only if  $(K, F) \in \mathcal{C}$  and  $K \in \text{Der}_0(\mathfrak{g})$  by Theorem 3.1. The next theorem gives the forms of  $\text{Der}_0(\mathfrak{g})$ , which are easy to be checked by Formula (9).

**Theorem 4.1.** *For Lie-Poisson structures (1)-(6), a pair  $(K, F)$  defines a quadratic deformation if and only if  $K$  has the following forms:*

$$(1) \quad K \in \mathfrak{sl}(3), \quad (2) \quad K \in \mathfrak{o}(3), \quad (3) \quad K \in \mathfrak{o}(2, 1),$$

$$(4) \quad \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix}, \quad (5) \quad \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix}, \quad (6) \quad \begin{pmatrix} 0 & 0 & 0 \\ \alpha & \delta & \theta \\ \beta & \gamma & -\delta \end{pmatrix}.$$

In all cases above,  $F$  may be any cubic polynomial such that  $\hat{K}F = 0$ .

For Case (1) in Theorem 2.3, the work of the classification has been done in [11]. For other cases, One needs to choose firstly a representative element in  $\text{Der}_0(\mathfrak{g})$  in each adjoint orbit of  $\text{Aut}(\mathfrak{g})$  given in Theorems 2.4 and 4.1 respectively and then to find compatible homogeneous cubic polynomials. Here we only write out two examples and leave the others to interested readers.

- The compatible pair of  $\mathfrak{o}(3)$  is  $(0, x^2 + y^2 + z^2)$  and  $\text{Der}_0\mathfrak{o}(3) = \text{Der}\mathfrak{o}(3) = \mathfrak{o}(3)$ . Then by Theorem 3.1 all the quadratic deformations determined by those  $(K, F)$  such that  $K \in \mathfrak{o}(3)$  and  $\hat{K}F = 0$ . When  $K = 0$ ,  $F$  may be any cubic polynomial. When  $K \neq 0$ , by Theorem 2.4 and Theorem 4.1, any quadratic deformation is isomorphic to one of the following forms:

$$K = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F = a(x^2 + y^2)z + bz^3, \quad (\alpha > 0, a \geq 0). \quad (17)$$

In fact, the automorphism group of  $\mathfrak{o}(3)$  is  $O(3)$  and any element of  $\mathfrak{o}(3)$  can be transformed to the above standard form via the adjoint action of  $O(3)$ . Moreover, the isotropy group of  $K$  can change  $z$  to  $-z$  so that we can take  $a \geq 0$ .



- The compatible pair of  $\mathfrak{o}(2, 1)$  is  $(0, x^2 + y^2 - z^2)$ . Similar to the discussion above, one can check that when  $K = 0$ ,  $F$  may be any cubic polynomial and when  $K \neq 0$ , any quadratic deformation is isomorphic to one of the following forms:

$$\begin{aligned}
(1) \quad K_1 &= \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_1 = a(x^2 + y^2)z + bz^3 \\
(2) \quad K_2 &= \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad F_2 = a(x - y)^3 + b(x - y)(x^2 + y^2 - z^2) \\
(3) \quad K_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha & 0 \end{pmatrix}, \quad F_3 = ax^3 + bx(y^2 - z^2)
\end{aligned}$$

In fact,  $\mathfrak{o}(2, 1)$  is isomorphic to  $\mathfrak{sl}(2, \mathbb{R})$  and any element of  $\mathfrak{o}(2, 1)$  can be corresponded to the one of above standard forms via the analysis the adjoint orbits of  $SL(2, \mathbb{R})$ . The forms of  $F_1$  and  $F_3$  are obvious.  $F_2$  can be checked after a coordinate transformation:  $\xi = \frac{1}{2}(x + y)$ ,  $\eta = \frac{1}{2}(x - y)$ .

## 5 Cases (7)-(10)

In this section we follow the scheme shown in Section 3. to classify quadratic deformations of other cases. First we consider the book algebra which is Case (7) listed in Theorem 2.3. To save space we just write out details for three Jordan standard forms only, given by (20), (22) and (23) respectively.

Note that the real projective plane  $P^2$  is isomorphic to a  $GL(3)$  homogenous space by dividing an isotropy subgroup keeping the subspace  $\mathbb{R}e_3$  invariant. Such subgroup is just  $(G_7 \times \mathbb{R}^\sharp I)$ , where  $\mathbb{R}^\sharp = \mathbb{R} - \{0\}$ . That is,

$$P^2 \cong (G_7 \times \mathbb{R}^\sharp I) \backslash GL(3) \cong G_7 \backslash GL(3) / \mathbb{R}^\sharp I,$$

Actually, such a correspondence is because any matrix of  $G_7$  preserves  $e_3$  then the multiplication with it preserves the last column of any matrix. Moreover, it is easy to see that  $G_7 G_K = G_K G_7$  and  $\mathbb{R}^\sharp I \subset G_K$  for any Jordan standard form  $K$ . This means that, for Case (7), the double orbit space (16) has the following form:

$$G_7 \backslash \mathcal{J}_K \cong (G_7 \backslash GL(3)) / G_K \cong (G_7 \times \mathbb{R}^\sharp I) \backslash GL(3) / G_K \cong P^2 / G_K. \quad (18)$$

For any  $[\alpha, \beta, \gamma] \in P^2$ , where  $v = (\alpha, \beta, \gamma)$  satisfies  $\|v\| = 1$  and  $(v, e_3) \geq 0$ , it corresponds an orthogonal matrix  $T$ . For  $v = e_3$ , let  $T = I$ . For  $(v, e_3) < 1$ , let

$$w = (e_3 - (v, e_3)v) / (\sqrt{1 - (v, e_3)^2}), \quad T = (w, v \times w, v)^T \in SO(3). \quad (19)$$

For a fixed Jordan standard form  $K$  with its isotropy subgroup  $G_K$  and a compatible pair  $(K, F) \in \mathcal{C}$  given in [11], let  $p_i$  denote a representative element of each orbit of  $P^2$  under the action of  $G_K$  and  $T_i$  is the corresponding orthogonal matrix given above. Then the pair  $K_i = T_i K T_i^{-1}$  and  $F_i = F \circ T_i^{-1}$  is also a compatible pair on each  $G_K$  orbit by (14) (Here  $\det(T_i) = 1$ ). First we take

$$K = \text{diag}(\lambda_1, \lambda_2, \lambda_3), \quad \lambda_1 + \lambda_2 + \lambda_3 = 0, \quad \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq 0. \quad (20)$$

Its compatible cubic polynomials are in the form  $F = axyz$  and  $G_K$  is the set of non-singular diagonal matrices.

**Lemma 5.1.** *For the Jordan form  $K$  given by (20),  $P^2/G_K$  contains seven orbits with following representative elements  $p_i$  and corresponding compatible pairs  $(K_i, F_i)$ :*

- (1)  $p_1 = [0, 0, 1]$ ,  $K_1 = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ ,  $F_1 = axyz$ .
- (2)  $p_2 = [0, 1, 0]$ ,  $K_2 = \text{diag}(\lambda_3, \lambda_1, \lambda_2)$ ,  $F_2 = axyz$ .
- (3)  $p_3 = [1, 0, 0]$ ,  $K_3 = \text{diag}(\lambda_3, \lambda_2, \lambda_1)$ ,  $F_3 = -axyz$ .
- (4)  $p_4 = [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0]$ ,  $K_4 = \frac{1}{2} \begin{pmatrix} 2\lambda_3 & 0 & 0 \\ 0 & -\lambda_3 & \lambda_1 - \lambda_2 \\ 0 & \lambda_1 - \lambda_2 & -\lambda_3 \end{pmatrix}$ ,  $F_4 = \frac{a}{2}x(z^2 - y^2)$ .
- (5)  $p_5 = [0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ ,  $K_5 = \begin{pmatrix} \frac{\lambda_2 + \lambda_3}{2} & 0 & \frac{\lambda_3 - \lambda_2}{2} \\ 0 & \lambda_1 & 0 \\ \frac{\lambda_3 - \lambda_2}{2} & 0 & \frac{\lambda_2 + \lambda_3}{2} \end{pmatrix}$ ,  $F_5 = \frac{a}{2}y(z^2 - x^2)$ .
- (6)  $p_6 = [\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}]$ ,  $K_6 = \begin{pmatrix} \frac{\lambda_1 + \lambda_3}{2} & 0 & \frac{\lambda_3 - \lambda_1}{2} \\ 0 & \lambda_2 & 0 \\ \frac{\lambda_3 - \lambda_1}{2} & 0 & \frac{\lambda_1 + \lambda_3}{2} \end{pmatrix}$ ,  $F_6 = \frac{a}{2}y(x^2 - z^2)$ .
- (7)  $p_7 = [\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}]$ ,  $K_7 = \begin{pmatrix} \frac{\lambda_1 + \lambda_2 + 4\lambda_3}{6} & \frac{\sqrt{3}(\lambda_2 - \lambda_1)}{6} & \frac{\sqrt{2}(2\lambda_3 - \lambda_1 - \lambda_2)}{6} \\ \frac{\sqrt{3}(\lambda_2 - \lambda_1)}{6} & \frac{\lambda_1 + \lambda_2}{2} & \frac{\sqrt{6}(\lambda_1 - \lambda_2)}{6} \\ \frac{\sqrt{2}(2\lambda_3 - \lambda_1 - \lambda_2)}{6} & \frac{\sqrt{6}(\lambda_1 - \lambda_2)}{6} & \frac{\lambda_1 + \lambda_2 + \lambda_3}{3} \end{pmatrix}$ ,  
 $F_7 = \frac{\sqrt{3}a}{9}z^3 + G(x, y, z)$ , where  $G(x, y, z)$  is cubic Polynomial without  $z^3$ .

**Proof.** Note that for each  $A = \text{diag}(m, n, p) \in G_K$ , then  $[\alpha, \beta, \gamma]A = [m\alpha, n\beta, p\gamma]$ . If  $\alpha = 0$ ,  $\beta = 0$ ,  $\gamma \neq 0$ , let  $p = \frac{1}{\gamma}$ , we obtain the first representative element  $p_1$ . The others are same.  $(K_i, F_i)$  can be get by (19) easily. ■

**Proposition 5.2.** *For Lie-Poisson structure  $\pi = \frac{1}{2}\widehat{I} \wedge \widehat{k}$  ( $k = (0, 0, 1)^T$ ) and  $K$  in form (20), then any compatible pair  $(\widetilde{K}, \widetilde{F}) \in \mathcal{C}_{k,0}$  such that  $\widetilde{K} \in \mathcal{J}_K$  is isomorphic to one of Pairs (1), (2), (3) listed in Lemma 5.1 with*

$$F_1 = \frac{1}{6}(\lambda_2 - \lambda_1)xyz, \quad F_2 = \frac{1}{6}(\lambda_1 - \lambda_3)xyz, \quad F_3 = \frac{1}{6}(\lambda_3 - \lambda_2)xyz.$$

**Proof.** We only give the proof of Case (1) and Case (4) in Lemma 5.1 and the proof of other cases is similar. First, for  $(k, f) = ((0, 0, 1)^T, 0)$  being the compatible pair of the book algebra, Equation (12) in this case is:

$$\frac{\partial F}{\partial z} = -\frac{1}{6}(-a_{21}x^2 + a_{12}y^2 + (a_{11} - a_{22})xy + a_{13}yz - a_{23}xz), \quad (21)$$

where  $K = (a_{ij})$ . Thus, for Case (1), one should has  $\frac{\partial F_1}{\partial z} = \frac{1}{6}(\lambda_2 - \lambda_1)xy$  by Formula (21). On the other hand we have  $F_1 = axyz$ , so if  $a = \frac{1}{6}(\lambda_2 - \lambda_1)$ , the equation is satisfied and  $(K_1, \frac{1}{6}(\lambda_2 - \lambda_1)xyz)$  makes quadratic deformation. As in Case (4), by Formula (21), we have

$$\frac{\partial F_4}{\partial z} = -\frac{1}{6}[(\lambda_3 - \frac{\lambda_1 + \lambda_2}{2})xy - \frac{\lambda_1 - \lambda_2}{2}xz].$$

We know that  $F_4 = \frac{a}{2}x(z^2 - y^2)$  so that  $\frac{\partial F_4}{\partial z} = axz$ . This implies that  $2\lambda_3 = \lambda_1 + \lambda_2 \implies \lambda_3 = 0$  since  $\lambda_1 + \lambda_2 + \lambda_3 = 0$ , this is a contradiction since we have  $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq 0$ , so for any  $a$ , Formula (21) couldn't be satisfied so that there is no quadratic deformation in this case. ■

Next we consider another Jordan form  $K$  whose corresponding compatible cubic polynomials(see [11]) and isotropy subgroup are as follows:

$$K = \text{diag}(\lambda, \lambda, -2\lambda), \quad F = mxyz + nx^2z + py^2z, \quad G_K = \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}, \quad (22)$$

where  $\lambda \neq 0$  and  $A \in GL(2)$ . It is easy to check that  $P^2/G_K$  has three orbits.

**Lemma 5.3.** *With same notation in Lemma 5.1 but  $K$  being form (22), then we have*

$$(1) \quad p_1 = [0, 0, 1], \quad K_1 = \text{diag}(\lambda, \lambda, -2\lambda), \quad F_1 = mxyz + nx^2z + py^2z.$$

$$(2) \quad p_2 = [0, 1, 0], \quad K_2 = \text{diag}(-2\lambda, \lambda, \lambda), \quad F_2 = mxyz + nxy^2 + pxz^2.$$

$$(3) \quad p_3 = [0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}], \quad K_3 = \begin{pmatrix} -\frac{1}{2}\lambda & 0 & -\frac{3}{2}\lambda \\ 0 & \lambda & 0 \\ -\frac{3}{2}\lambda & 0 & -\frac{1}{2}\lambda \end{pmatrix},$$

$$F_3 = \frac{m}{2}y(x^2 - z^2) + \frac{\sqrt{2}}{2}ny^2(z + x) + \frac{\sqrt{2}}{4}p(z - x)^2(z + x).$$

**Proposition 5.4.** *With same notation in Prop.5.2 but  $K$  being form (22), then any compatible pair  $(\tilde{K}, \tilde{F}) \in \mathcal{C}_{k,0}$  such that  $\tilde{K} \in \mathcal{J}_K$  is isomorphic to one of Pairs (1), (2) listed in Lemma 5.3 with  $F_1 = 0$  and  $F_2 = \frac{1}{2}\lambda xyz + nxy^2$  respectively.*

Finally we consider the following Jordan form  $K$  whose corresponding compatible cubic polynomials(see [11]) and isotropy subgroup are as follows:

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad F = pz^3 + 2qz^2x - qy^2z, \quad G_K = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}. \quad (23)$$

In this case it is easy to check that  $P^2/G_K$  has three orbits and two of them can make quadratic deformations.

**Lemma 5.5.** *With same notation in Lemma 5.1 but  $K$  being form (23), then we have*

$$\begin{aligned} (1) \quad p_1 &= [0, 0, 1], \quad K_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_1 = pz^3 + 2qz^2x - qy^2z. \\ (2) \quad p_2 &= [0, 1, 0], \quad K_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_2 = px^3 + 2qx^2y - qz^2x. \\ (3) \quad p_3 &= [1, 0, 0], \quad K_3 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad F_3 = px^3 + 2qx^2z - qy^2x. \end{aligned}$$

**Proposition 5.6.** *With same notation in Prop.5.2 but  $K$  being form (23), then any compatible pair  $(\tilde{K}, \tilde{F}) \in \mathcal{C}_{k,0}$  such that  $\tilde{K} \in \mathcal{J}_K$  is isomorphic to one of Pairs (2), (3) in Lemma 5.5 with  $F_2 = px^3 - \frac{1}{6}x^2y + \frac{1}{12}z^2x$  and  $F_3 = px^3 - \frac{1}{6}x^2z + \frac{1}{12}y^2x$ .*

For Lie-Poisson structures (8)-(10), by Theorem 2.4, we know that  $G_l \subset G_7$  so that  $G_l$ -orbit  $\subset G_7$ -orbit ( $l = 8, 9, 10$ ) in  $\mathcal{J}_K$ . Therefore, for these cases, the double orbit space (16) has the following form by (18):

$$G_l \backslash \mathcal{J}_K \cong (G_l \backslash G_7) \cdot (G_7 \backslash \mathcal{J}_K) \cong (G_l \backslash G_7) \cdot (P^2/G_K).$$

It is easy to see that the quotient spaces  $G_l \backslash G_7$  ( $l = 8, 9, 10$ ) are 2-dim. manifolds and their representative matrices can be given explicitly by fact that any invertible matrix can be decomposed into product of an orthogonal matrix with a lower triangular matrix (with a symmetric matrix for Case (9)). For example,

$$G_{10} \backslash G_7 \cong S^1 \times \mathbb{R}^\# \cong \{ T_\alpha Q_s = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix}, s \neq 0 \}.$$

Thus, for Case (10), one has  $G_{10} \backslash \mathcal{J}_K \cong (S^1 \times \mathbb{R}^\sharp) \cdot (P^2/G_K)$ . Now let  $K$  be given in (20) and analyze Case (10). From Lemma 5.1, we know that  $\mathcal{J}_K$  splits into seven  $G_7$ -orbits, i.e.,

$$\mathcal{J}_K = \bigcup_{i=1}^7 (G_7 \cdot K_i) \Rightarrow G_{10} \backslash \mathcal{J}_K = \bigcup_{i=1}^7 G_{10} \backslash (G_7 \cdot K_i) \cong \bigcup_{i=1}^7 (S^1 \times \mathbb{R}^\sharp) \cdot K_i$$

Note that  $Q_s K_i = K_i Q_s$  and  $sF_i \circ Q_s^{-1} = F_i$  for  $i = 1, 2, 3, 5, 6$  in Lemma 5.1. Consequently, all  $G_{10}$ -orbits in  $\mathcal{J}_K$ ,  $(S^1 \times \mathbb{R}^\sharp) \cdot K_i$ , can be parameterized by  $i, \alpha$  and  $(i = 4, 7) s$ . Moreover, by (14) and fact that  $\det(T_\alpha Q_s) = s$ , we can get their representative compatible pairs:

$$(T_\alpha K_i T_\alpha^{-1}, F_i \circ T_\alpha^{-1}), \quad i = 1, 2, 3, 5, 6, \quad \alpha \in [0, 2\pi)$$

$$((T_\alpha Q_s) K_i (T_\alpha Q_s)^{-1}, sF_i \circ (T_\alpha Q_s)^{-1}), \quad i = 4, 7, \quad s \neq 0, \quad \alpha \in [0, 2\pi).$$

The next step is to figure out all representative compatible pairs satisfying Equation (12) from those given above to classify quadratic deformations of Lie-Poisson structure Case (10) on  $\mathcal{J}_K$ . It is easy to check that there is no quadratic deformation for  $i = 1, 2, 3$  listed in Lemma 5.1.

As the last example, we take  $K$  as given in (22). In this case we know that  $\mathcal{J}_K$  splits into three  $G_7$ -orbits with their representative compatible pairs  $(K_i, F_i)$ ,  $i = 1, 2, 3$  listed in Lemma 5.3. Here we only write out the conclusion for  $K_1 = K$ .

**Proposition 5.7.** *For Lie-Poisson structure Case (10) and  $K$  being (22), then any quadratic deformation  $(\tilde{K}, \tilde{F})$  such that  $\tilde{K} \in G_7 \cdot K$  is isomorphic to  $(K, -2\lambda x^2 z)$ .*

**Proof.** It is obviously that  $G_7 \cdot K = G_{10} \cdot K$  in this case since  $(T_\alpha Q_s) K (T_\alpha Q_s)^{-1} = K$  so that we can take  $(K_1, F_1)$  given in Lemma 5.3 for  $i = 1$  as the representative compatible pair on this orbit, which satisfies Equation (12) if and only if  $\frac{\partial F_1}{\partial z} = -2\lambda x^2$ . Thus when  $m = p = 0$ ,  $n = -2\lambda$ , the equation is satisfied so that this pair makes quadratic deformation. ■

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